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Swinging Atwood Machine. Far- and Near-Resonance Region

A. Roy Chowdhury[1] and M. Debnath[1]

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The swinging Atwood machine, a prototype nonlinear dynamical system, is analyzed following an idea of Bogoliubov and Mitropolsky. A series solution is found for the radial and angular displacement as functions of time. The analysis is repeated in the resonance case, when the frequency of the driving force maintains a fixed ratio to that of the free motion. The condition of resonance requires the mass ratio μ to be equal to $2j^2-1$, where j is an integer not equal to one.

1. INTRODUCTION

The analysis of nonlinear dynamical systems is of utmost importance in various branches of modern science (Feigenbaum, 1982), and many techniques have been discovered to analyze and study such problems (Guckenheimer and Holmes, 1983). Nonlinearity appears at the very root of classical dynamics, a simple example being the case of a pendulum. Another classic nonlinear system is the swinging Atwood machine initially discussed by Tufillaro *et al.* (1984; Tufillaro, 1985). Here we present a detailed analysis and explicit solution of such a system following an idea of Bogoliubov and Mitropolsky (1961). The cases of resonance and nonresonance are discussed separately and the possible and admissible values of the mass ratio $(m/M)^{-1}$ are obtained from the condition of resonance.

2. FORMULATION

The basic equations of motion for the swinging Atwood machine in polar coordinates (r, θ) can be written as

$$\begin{aligned}(1+\mu)\ddot{r} &= r\dot{\theta}^2 + g(\cos\theta - \mu)\\ r\ddot{\theta} &= -2\dot{r}\dot{\theta} - g\sin\theta\end{aligned} \tag{1}$$

[1]High Energy Physics Division, Department of Physics, Jadavpur University, Calcutta 700032, India.

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where θ is the inclination to the vertical and μ is the mass ratio $(=M/m)$. To convert it in an autonomous form, we set

$$\dot{\theta}=y; \qquad \dot{r}=x$$

whence

$$\begin{aligned} \frac{dx}{d\theta}&=\frac{r}{1+\mu}y+g\frac{\cos\theta-\mu}{y(1+\mu)} \\ \frac{dy}{d\theta}&=-\frac{2x}{r}-g\frac{\sin\theta}{ry} \end{aligned} \tag{2}$$

Eliminating x from these, we arrive at a second-order equation for y:

$$\frac{d^2y}{d\theta^2}+\frac{2}{1+\mu}y=gf\left(\theta, y, \frac{dy}{d\theta}\right) \tag{3}$$

where

$$f\left(\theta, y, \frac{dy}{d\theta}\right)=-\frac{3+\mu}{r(1+\mu)}\frac{\cos\theta}{y}+\frac{2\mu}{r(1+\mu)}\frac{1}{y}+\frac{\sin\theta}{r}\frac{1}{y^2}\frac{dy}{d\theta}$$

Equation (3) actually describes the case of a perturbed harmonic motion with frequency $w^2=2/(1+\mu)$, when $g=0$. Equation (3) has a solution of the form

$$y=a\cos\psi; \qquad \psi=w\theta+\varphi \tag{4}$$

Following (4), we seek a solution of the perturbed system in the form

$$y=a\cos\psi+gu_1(a,\psi,\theta)+g^2u_2(a,\psi,\theta)+O(g^3)+\cdots \tag{5}$$

whence the equations of modulated amplitude a and phase ψ are

$$\begin{aligned} \frac{da}{d\theta}&=gA_1(a)+g^2A_2(a)+\cdots \\ \frac{d\psi}{d\theta}&=w+gB_1(a)+g^2B_2(a)+\cdots \end{aligned} \tag{6}$$

The functions $A_1(a)$, $B_1(a)$, and u_1 are constructed from the Fourier expansions of f occurring on the right-hand side of equation (3),

$$\begin{aligned} A_1(a)&=-\frac{1}{4\pi^2w}\int_0^{2\pi}\int_0^{2\pi}f_0(a,\psi,\theta')\sin\psi\,d\theta'\,d\psi \\ &=0 \end{aligned} \tag{7}$$

For the nonresonant case, B_1 is given as

$$B_1(a) = -\frac{1}{4\pi^2 wa} \int_0^{2\pi} \int_0^{2\pi} L\frac{\cos\theta'}{\cos\psi} + M\frac{1}{\cos\psi}$$

$$+ N\frac{\sin\theta' \sin\psi}{\cos^2\psi} \cos\psi \, d\theta' \, d\psi$$

$$= -\frac{2\mu}{(1+\mu) r wa^2} \tag{8}$$

where

$$L = -\frac{3+\mu}{ar(1+\mu)}; \qquad M = \frac{2\mu}{ar(1+\mu)}; \qquad N = -\frac{w}{ar}$$

On the other hand, the expression for u_1 is (Bogoliubov and Mitropolsky, 1961)

$$u_1(a, \psi, \theta') = \frac{1}{2\pi^2} \sum_{n,m} \left\{ \frac{\cos(n\theta' + m\psi)}{w^2 - (n\nu + mw)^2} I_1 + \frac{\sin(n\theta' + m\psi)}{w^2 - (n\nu + mw)^2} I_2 \right\} \tag{9}$$

$$I_{1,2} = \int_0^{2\pi} \int_0^{2\pi} f_0(a, \psi, \theta') \begin{Bmatrix} \cos(n\theta' + m\psi) \\ \sin(n\theta' + m\omega) \end{Bmatrix} d\theta' \, d\psi \tag{10}$$

with $\omega^2 \neq (n\nu + m\omega)^2$. Keeping terms up to a few harmonics, we get

$$u_1(a, \psi, \theta') = \frac{M}{2w^2} \cos 3\psi + \frac{2(L+N)}{w^M - (\nu + w)^2} \cos(\theta' + \psi)$$

$$+ \frac{2(L-N)}{w^2 - (\nu - w)^2} \cos(\theta' - \psi) - \frac{2(L+3N)}{w^2 - (\nu + 3w)^2} \cos(\theta' + 3\psi)$$

$$+ \frac{2(3N - L)}{w^2 - (\nu - 3w)^2} \cos(\theta' - 3\psi) \tag{11}$$

In our present case $\nu = 1$.

So the corresponding solution of (6) up to first order in g is

$$a = \text{const.} = a_0, \qquad \psi = \left(w - \frac{2\mu g}{(1+\mu) r w a_0^2} \right) \theta \tag{12}$$

whence

$$y = a_0 \cos\psi + g u_1(a, \psi, \theta) + O(g^2)$$

with u_1 given in (11).

3. RESONANCE CASE

Consider the case when the driving term frequency is proportional to the original frequency w, that is (Strubble, 1974),

$$w \approx p/q \tag{13}$$

(p, q) being prime numbers. Near the resonance region we set

$$w^2 = [(p/q)\nu]^2 + g\Delta \tag{14}$$

The condition for resonance leads to

$$nq + (m \pm 1)p = 0$$

whence

$$A_1(a, \varphi) = -\frac{q}{4\pi^2 \nu p} \sum_{\sigma} e^{iq\sigma\varphi} J_1(\sigma)$$

$$J_1(\sigma) = \int_0^{2\pi} \int_0^{2\pi} f_0(a, \psi, \theta') e^{-iq\sigma\varphi'} \sin\theta' \, d\theta' \, d\psi \tag{15}$$

and finally

$$B_1(a, \varphi) = \frac{\Delta}{2} \frac{q}{p\nu} \frac{q}{4\pi^2 a p \nu} \sum_{\sigma} e^{iq\sigma\varphi} J_2(\sigma)$$

$$J_2(\sigma) = \int_0^{2\pi} \int_0^{2\pi} f_0(a, \theta', \varphi) e^{-iq\sigma\varphi'} \cos\psi \, d\theta' \, d\psi \tag{16}$$

$$\varphi = \psi - \frac{p}{q} \nu\theta$$

Evaluating the elementary trigonometric integrals (for $\sigma = 1$, others being zero; also, we have set $q = 2$), we obtain

$$\begin{aligned} A_1(a, \varphi) &= -\frac{iq}{4\pi^2 \nu p} e^{iq\varphi} \left\{ \frac{2\pi^2(3+\mu)}{ar(1+\mu)} - \frac{4\pi^2 w}{ar} \right\} \\ &= -\frac{2i\pi^2}{4\pi^2} e^{2i\varphi}(2N - L) \\ &= +i(L - 2N)e^{2i\varphi} \end{aligned} \tag{17}$$

Also, $B_1(a, \varphi)$ turns out to be

$$B_1(a, \varphi) = \Delta + \frac{w}{a^2 r} e^{2i\varphi} \tag{18}$$

So, up to first order in g, the equations for amplitude and phase turn out to be

$$\frac{da}{d\theta}=gA_1(a,\theta,\varphi); \qquad \frac{d\varphi}{d\theta}=gB_1(a,\theta,\varphi) \tag{19}$$

Using the expressions for A_1 and B_1, we eliminate a and obtain the following equation for φ:

$$\frac{d^2\varphi}{d\theta^2}+\left(\frac{d\varphi}{d\theta}\right)^2\left(-2i-\frac{\beta r}{gw}\right)+\frac{d\varphi}{d\theta}\left(2i\Delta g+2g\Delta\frac{\beta r}{gw}\right)$$
$$=-g^2\Delta^2\frac{\beta r}{grw} \tag{20}$$

We now set $d\varphi/d\theta=Z$. Then (20) is converted to

$$\dot{Z}+Z^2K_3-ZK_2-K_1=0 \tag{21}$$

with $K_1=-2i-\beta r/gw$, $K_3=-g\Delta^2\beta r/w$, and $K_2=2i\Delta g+2\Delta g\beta r/w$. Integrating (19), we arrive at

$$\varphi=\frac{-C_1\ln\cosh[C_1\theta+\tanh^{-1}(m'/2C_1)]-m'C_1\theta}{C_1^2-m'^2/4} \tag{22}$$

with $C_1=(m'^2/4+n')^{1/2}$, $m'=K_2/K_3$, and $n'=K_1/K_3$. Once φ is explicitly determined by (22), a is known via

$$a^2=-\frac{gw}{r}\frac{e^{2i\varphi}}{\varphi-g\Delta}$$

Finally, it is interesting to observe that the condition of resonance leads to

$$\frac{p}{q}=\left(\frac{2}{1+\mu}\right)^{1/2} \qquad \text{if} \quad p=1$$

Then

$$\mu=2q^2-1 \qquad (q\neq 1). \tag{23}$$

Equation (23) yields the possible physical values of μ, the mass ratio occurring in the Atwood machine.

4. CONCLUSION

In the above analysis we found explicit solutions for the amplitude and phase up to first order in g, considering separately the cases of resonance. It is an interesting outcome of this analysis that the mass ratio, which was arbitrary to start with, becomes determined and is equal to $2j^2-1$, j an integer.

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